Qualifying Exam Problems (Quantum Mechanics)

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1. Consider a two-dimensional vector space which is spanned by |1> and |2> basis vectors. There exists a Hermitian operator Ω satisfying the followings:

$<1|\Omega||1> = 0$, $<1|\Omega|2> = 1$, $<2|\Omega|1> = 1$, $<2|\Omega|2> = 0$

(1) What is a matrix representation of Ω in the above basis?

(2) Calculate the eigenvalues and the corresponding eigenvectors of Ω using the result of (1). Also, express the eigenvectors in terms of the basis kets.

(3) We use the eigenvectors of Ω as the new basis vectors. Calculate the transformation matrix U from the old basis to the new basis. Prove that $U^{+}\Omega U$ is a diagonal matrix whose diagonal elements are the eigenvalues of Ω.

2. The Hamiltonian operator H is given as follows.

$H = \Delta|\alpha><\alpha| + \Delta|\alpha><\alpha'| + \Delta|\alpha'><\alpha'| + \Delta|\alpha'><\alpha|$ where $|\alpha>$, $|\alpha'>$ are the eigenvectors of a Hermitian operator A and Δ is a positive real number. α and α' also represent the corresponding eigenvalues of A ($\alpha \neq \alpha'$).

(1) Write down the matrix representation of H in the eigenbasis of A. Are $|\alpha>$ and $|\alpha'>$ the eigenvectors of the Hamiltonian?

(2) Calculate the eigenvalues and the corresponding eigenkets of the Hamiltonian H.

(3) A was measured to be α and then the energy was measured to be some eigenvalue of H. Finally A was measured again. What is the probability of measuring α' this time?

3. A particle of mass m moves in one dimension in a potential given by $V(x) = -a\delta(x)$, where $\delta(x)$ is the Dirac delta function. The particle is bound. Find the value x_0 such that the probability of finding the particle with $|x| < x_0$ is exactly 1/2.

4. The quantum states $|\alpha>$ and $|\beta>$ are given as follows.

$$|\alpha> \ = \frac{1}{\sqrt{2}}|E_1> + \frac{1}{\sqrt{2}}|E_2>, \ |\beta> \ = \frac{1}{\sqrt{2}}|E_1> - \frac{1}{\sqrt{2}}|E_2>$$

where $|E_1>$ and $|E_2>$ are the eigenkets of the Hamiltonian operator H such that $H|E_1> = E_1|E_1>$, $H|E_2> = E_2|E_2>$, and $E_1 > E_2$.

(1) At t=0, the initial state ket is given by $|E_1>$. Write down the state vector at t.

(2) At t=0, the initial state ket is given by $|\alpha>$. Calculate $|\alpha, t>$ and the transition probability $|<\alpha|\alpha, t>|^2$. Estimate the time interval after which the state significantly differs from the initial state.

5. A particle with mass m is in the following potential:

$V(x) = 0$ for $-L \le x \le 0$

$V(x) = V_0$ otherwise

(1) Calculate the eigenenergies and the eigenfunction in the position basis when $V_0 = \infty$.

(2) When V_0 is finite and positive and $E < V_0$, answer the following questions.

(a) Plot the ground state wavefunction roughly.

(b) Suppose that the width of the potential well is doubled. Explain how the ground state energy will change.

(3) If $L \to \infty$ and $E < V_0$, explain the wavefunctions in each region qualitatively.

6. A particle is incident on the step potential from the left side as follows: $V(x) = 0$ for $x \le 0$, $V(x) = V_0$ for $x \ge 0$

The wavefunction of the incident wave is given by $\psi_I = Ae^{ikx}$ $(E = \frac{\hbar^2 k^2}{2m})$

(1) When $E>V_0>0$, calculate the probabilities of reflection and transmission as a function of V_0.

(2) When $E<V_0$, calculate the probabilities of reflection and transmission as a function of V_0.

(3) When $V_0<0$, explain how the reflected and transmitted waves

behave.

7. Consider a particle in one dimension bound to a fixed center by a δ function potential of the form: $V(x) = -V_0\delta(x)$ ($V_0 > 0$)

(1) Write down energy eigenfunctions for x>0 and x<0 respectively.

(2) Apply boundary conditions at x = 0 and normalize the energy eigenfunction.

(3) Integrate the Schrodinger equation from -ε to +ε. Using the previous result, calculate the energy eigenvalue.

8. Consider a particle with a positive energy E in a δ function potential of the form: $V(x) = D\delta(x)$ $(D \neq 0)$

(1) Suppose that an incident wave is propagating from $x = -\infty$. Write down energy eigenfunctions for x>0 and x<0, respectively.

(2) Calculate the transmission probability as a function of E by applying boundary conditions at x=0.

9. The Hamiltonian of a simple harmonic oscillator, the energy eigenstates and the energy eigenvalues are given as follows:

$$H = \frac{p^2}{2m} + \frac{kx^2}{2} = \hbar\omega(a^+a + \frac{1}{2}) = \hbar\omega(n + \frac{1}{2})$$

$$H|n> = \hbar\omega(n + \frac{1}{2})|n> \quad \text{where } n=0,1,2,...$$

The a and a^+ operators can be written in position basis as follows:

$$a = \frac{1}{\sqrt{2}}(y + \frac{d}{dy}), \ a^+ = \frac{1}{\sqrt{2}}(y - \frac{d}{dy}) \ \text{ where } y = \sqrt{\frac{m\omega}{\hbar}}x$$

(1) Show that the Hamiltonian is parity-invariant.

(2) Using $a|0> = 0$, calculate the ground state wavefunction. You do not have to normalize the wavefunction.

(3) From (1) and (2), calculate parity eigenvalues of $|n>$ state.

10. The Hamiltonian of a simple harmonic oscillator, the energy eigenstates and the energy eigenvalues are given as follows:

$$H = T + V = \frac{p^2}{2m} + \frac{1}{2}kx^2 = \hbar\omega(a^+a + \frac{1}{2}) = \hbar\omega(n + \frac{1}{2})$$

$H|n> = \hbar\omega(n+\frac{1}{2})|n>$ where n=0,1,2,3....

The a and a^+ operators can be written in terms of x and p operators as follows:

$$a = \sqrt{\frac{m\omega}{2\hbar}}x + i\sqrt{\frac{1}{2m\omega\hbar}}p, \quad a^+ = \sqrt{\frac{m\omega}{2\hbar}}x - i\sqrt{\frac{1}{2m\omega\hbar}}p$$

(1) Calculate $<m|x|n>$, $<m|p|n>$, $<m|x^2|n>$, and $<m|p^2|n>$.

(2) Using the results of (1), prove that $<T> = <V>$ (virial theorem).

(3) Calculate $\Delta x \Delta p$ for the ground state.

11. The expectation value of an observable A in the state ψ is defined as follows

$$<A> = \int dx \psi^* A \psi.$$

(1) Assume that A does not vary with time. Show that the time variation of $<A>$ is given by

$$\frac{d}{dt}<A> = \frac{1}{i\hbar}<[A,H]>.$$

(2) Consider an one-dimensional motion of a particle of mass m in a potential $V(x)$ represented by the state wave function $\psi(x,t)$. Prove that the time variation of the expectation values of position and momentum are given by

$$\frac{d}{dt}<x> = <p>/m, \text{ and } \frac{d}{dt}<p> = -<\frac{dV}{dx}>.$$

(3) Explain the physical significance of these results.

12. Consider two uncoupled oscillators. The following creation and destruction operators can be defined respectively:

$N_1 = a_1^+ a_1, \quad [a_1, a_1^+] = 1, \quad [N_1, a_1^+] = a_1^+, \quad [N_1, a_1] = -a_1$

$N_2 = a_2^+ a_2, \quad [a_2, a_2^+] = 1, \quad [N_2, a_2^+] = a_2^+, \quad [N_2, a_2] = -a_2$

We assume that any pair of operators from different oscillators commute. We can obtain simultaneous eigenket of N_+ and N_- as follows: $|n_1, n_2\rangle = \frac{(a_1^+)^{n_1}(a_2^+)^{n_2}}{\sqrt{n_1! n_2!}}|0,0\rangle$

If we define the following operators,

$J_+ \equiv \hbar a_1^+ a_2, \quad J_- \equiv \hbar a_2^+ a_1, \quad J_z = \frac{\hbar}{2}(N_1 - N_2)$

(1) Prove that $[J_z, J_\pm] = \pm\hbar J_\pm$, $[J_+, J_-] = \pm 2\hbar J_z$.

(2) Prove that $J^2 \equiv J_z^2 + \frac{1}{2}(J_+J_- + J_-J_+) = (\frac{\hbar^2}{2})N(\frac{N}{2}+1)$ where $N \equiv N_1 + N_2$.

(3) From (1) and (2), J operators satisfy the same relations as angular momentum operators. What are the physical interpretations?

13. Particles are subjected to an infinite potential well.

$V(x) = 0 \quad \text{for} \quad 0 \le x \le L$

$V(x) = \infty \quad \text{otherwise}$

The energy eigenfunctions and energy eigenvalues of each particle are given as follows:

$\psi_n = \sqrt{\frac{2}{L}}\sin(\frac{n\pi x}{L}), \quad n = 1,2,3,\ldots, \qquad E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$

(1) Consider a system consisting of three electrons.

(a). What is the ground state energy of the system?

(b). If the energy of the system is measured to be $7\pi^2\hbar^2/mL^2$, write down the wavefunction of the system.

(2) Consider a system consisting of three pions. Pions are bosons with spin 0.

(a). What is the ground state energy of the system?

(b). If the energy of the system is measured to be $7\pi^2\hbar^2/mL^2$, write down the wavefunction of the system.

(3) Consider a system consisting of two electrons and a proton.

(a). What is the ground state energy of the system?

(b). If the energy of the system is measured to be $7\pi^2\hbar^2/mL^2$, write down the wavefunction of the system.

In this problem, the interactions between particles are neglected.

14. N identical non-interacting particles are subjected to a one dimensional simple harmonic potential.

$$H=\frac{p^2}{2m}+\frac{1}{2}kx^2$$

(1) What is the ground state energy of the system if they are spin 1/2 particles? There are two independent spin states (spin-up and spin-down) for each energy eigenstate.

(2) What is the ground state energy of the system if they are spin 0 particles?

15. The ground state wave function of a hydrogen atom reads:

$$\psi_{100}(\vec{r})=\frac{1}{\sqrt{\pi a_0^3}}\exp(-r/a_0).$$

and its energy is given by $E_0=-e_0^2/2a_0$, where a_0 is the Bohr radius. Compute the probability that the electron in a ground state of a hydrogen atom will be found at a distance from the nucleus greater than its energy would permit on the classical theory. (Coulomb energy is given by $V(r)=-\frac{e_0^2}{r}$.)

16. The Hamiltonian of a particle is given by $H=\frac{p^2}{2m}+V(r)$

(1) Prove that the Hamiltonian is parity-invariant.

(2) Prove that the Hamiltonian is rotationally invariant.

(3) From the symmetries of (1) and (2), write down the corresponding conserved quantities and discuss the physical implications.